1 Introduction - Buffon's Needle

In the 18th century, Count Buffon discovered a Monte Carlo (i.e., random) method for estimating the value of π: namely, one throws a needle of length L at a grid of parallel lines evenly spaced with distance D between them. After many trials, the proportion $\hat{p}$ of trials in which the needle crossed a line in the grid almost surely converges to the theoretical probability p of such a crossing in any single trial, from which one can "solve for π" in the formula $\frac{2L}{\pi D} = p \approx \hat{p}$. (That is, $\pi \approx \frac{2L}{D\hat{p}}$.)

Recently, interest has renewed in Buffon's needle problem for other sets – particularly, Hausdorff one-dimensional sets. The so-called Sierpinski's Gasket, $\mathcal{G}$, forms an example of such a set of recent interest - at each stage, one cuts off the three corners of each triangle and discards the rest, leaving triangles that are a third as long on each side. One does this endlessly, calling the set $\mathcal{G}_n$ at each stage, and calling the resulting fractal in the limit $\mathcal{G}$.

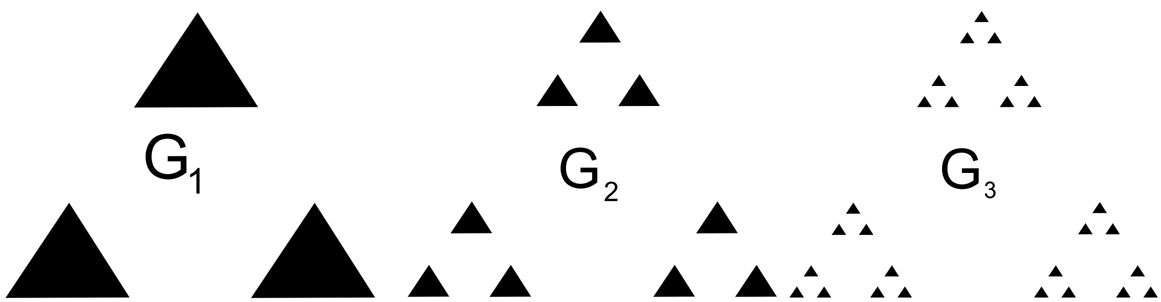

Figure 1: Three stages in the construction of a fractal, Sierpinski's gasket.

$\mathcal{G}$ is like a line in that both are **one-dimensional** - it can always be contained in 3^n balls of radius $(1/3)^n$ just like a line segment can. The Gasket is dispersed like a fractal, highly irregularly. There are two consequences of this: One, the **Buffon needle probability** of $\mathcal{G}$ is 0. Two, the **analytic capacity** of $\mathcal{G}$ is 0, meaning that $\mathcal{G}$ is never large enough to contain all of the singularities and large values of a (non-constant) bounded analytic function.

These facts are related, but neither group of problems is entirely understood. The applicant and his advisor, Alexander Volberg, have written papers on two Buffon's needle problems [2], [3], [4]. In [2], it was shown that $\mathcal{G}_n$ has Buffon needle probability less that a constant times $\frac{1}{n^{1/13}}$. An upper bound of this type was proved for other sets - fractal product sets having a "tiling direction" (on which copies of the projected shadow can be used to tile the line) around the same time by the supervising scientist, Izabella Łaba, together with Kelan Zhai in [9]. This case is important, in that it appears that tiling and non-tiling directions are the structured and pseudo-random cases, respectively, and that a satisfactory resolution of Buffon's needle problem in some more general cases is likely to result soon if all of these ideas are pushed further. The applicant and the supervising scientist are in an ideal position to resolve these problems in the near future. Other problems in analysis, like the boundedness of certain maximal operators that act by taking averages along curves, are also connected to Buffon's needle problem, as well as the Kakeya conjecture and problems in incidence geometry.

2 Differentiation theorems

Another problem regarding the geometric properties of low-dimensional fractals takes the form of a recent differentiation theorem of the sponsoring scientist and Malabika Pramanik, [7]. A fundamental and well-known result is the **Lebesgue differentiation theorem**, which states that the average of a function on small intervals shrinking to a point converges to the function's value at the center point for "almost every" (in the sense of Lebesgue measure) choice of starting point; one can say that intervals have the **differentiation property**. One needs only that the function has well-defined "averages" the first place. What [7] does is construct a certain random fractal set S on the real line so that the averages on small intervals may instead be taken on rescaled copies of the fractal, and again these averages converge to the function's value at the initial point for almost every choice of starting point.

The applicant and the sponsoring scientist would like to apply the ideas of [7] to another setting, that of Brownian paths. Brownian motion can be understood as a random walk (in one, two, three, or more dimensions) in which one takes a large number of very small steps, resulting in a random fractal curve. Such random sets in mathematics often prove that certain features of mathematical sets are "typical" even though it is hard to construct an example directly. It may be the case that Brownian paths have the differentiation property. If such a differentiation result holds in higher dimensions, then this would show that sets of low dimension can still differentiate functions in higher dimensions - in particular, sets of lower dimension than any current example.

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