Non-standard Analysis: Calculus of Infinitesimals

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Brief history of Calculus

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~~The end.~~ Abraham Robinson rigorously invents extended real number line including infinitesimals and infinities in 1960.

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$\mathbf{q}, \mathbf{r} \in \hat{\mathbb{Q}}$, let $\mathbf{q} \sim \mathbf{r}$ **iff** $\forall \varepsilon > 0$, $|q_i - r_i| < \varepsilon$ for all $i > N$ large enough.

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$\mathbb{R} = \hat{\mathbb{Q}} / \sim$ is the unique complete ordered field. $q \to (q, q, q, q, ...)$ is an (isometric) injection $\mathbb{Q} \to \mathbb{R}$.

How do we make numbers into more numbers?

Use equivalence classes!

Foundation of non-standard analysis: Consider the **real** numbers, $\mathbb{R}$. Let $\hat{\mathbb{R}}$ be the set of ~~Cauchy~~ **all** sequences $(r_1, r_2, \ldots)$, $r_i \in \mathbb{R}$.

Define some other equivalence relation $\sim$.

$\mathcal{R} = \hat{\mathbb{R}} / \sim$ is ~~the unique complete ordered field~~ $\mathcal{R}$ the field of so-called **hyperreal numbers**. $x \to (x, x, x, x, \ldots)$ is an (isometric*) injection $\mathbb{R} \to \mathcal{R}$.

*: $\mathcal{R}$ isn't a metric space in the sense of there being a **real-valued** metric on it.

What kind of equivalence relation?

Natural operations on $\hat{\mathbb{R}}$:

$$(x_1, x_2, x_3, ...) \cdot (y_1, y_2, y_3, ...) = (x_1 y_1, x_2 y_2, x_3 y_3, ...)$$

$$(x_1, x_2, x_3, ...) + (y_1, y_2, y_3, ...) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, ...)$$

We want $\mathcal{R}$ to be a field. Consider:

$$(0, 1, 0, 1, 0, 1, ...) \cdot (1, 0, 1, 0, 1, 0) = \mathbf{0}.$$

One of the left-hand terms must be $\sim \mathbf{0}$.

Filter axioms - “large subsets” of $\mathbb{N}$

$\mathbf{x} \sim \mathbf{y}$ **iff** $\{n \in \mathbb{N} : x_i = y_i \in \mathcal{F}\}$, where $\mathcal{F}$ is a special “filter” on $\mathbb{N}$.
Let $\mathcal{F} \subset \mathcal{P}(X)$ be a **filter** on X if:
1) $\emptyset \notin \mathcal{F}$
2) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
3) $A \in \mathcal{F}$ and $B \supseteq A \Rightarrow B \in \mathcal{F}$
4) $\mathcal{F}$ is **free** if $\mathcal{F} \neq \{A : x_0 \in A\}$
5) $\mathcal{F}$ is an **ultrafilter** if $\forall A \in \mathcal{P}(X)$, $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

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2) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$
3) $A \in \mathcal{F}$ and $B \supseteq A \Rightarrow B \in \mathcal{F}$ **1-3 define an equivalence relation on** $\hat{\mathbb{R}}$.
4) $\mathcal{F}$ is **free** if $\mathcal{F} \neq \{A : x_0 \in A\}$ **Implies that** $\mathcal{R} \neq \mathbb{R}$
5) $\mathcal{F}$ is an **ultrafilter** if $\forall A \in \mathcal{P}(X)$, $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$. **Implies that** $<, >, =$ **trichotomy extends naturally to** $\mathcal{R}$.

Free ultrafilters - they exist?

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Free ultrafilter $\mathcal{F}$: Exists via Zorn's Lemma construction. $\mathcal{F} \supset \mathcal{F}_r$. Maximize $\mathcal{F}$.

Interlude


Why settle for STANDARD?
When the Choice is mine, I go with
Robinson Free Ultrafilters.
Analysts agree...
More robust than
Fréchet Filters
Robinson
Free
Ultrafilters
CIGARETTES

Now we're in business!

For example, $\omega = (1, 2, 3, 4, 5, ...)/ \sim = [n]_{\mathcal{F}}$ is an "infinite number" while $1/\omega := (1, 1/2, 1/3, 1/4, ...) = [1/n]_{\mathcal{F}} = \varepsilon$ is an "infinitesimal." $\omega \cdot \varepsilon = (1, 1, 1, 1, ...) = [1]_{\mathcal{F}}$.

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For $x \in \mathbb{R}$, ${}^*x := (x, x, x, x, x, ...)/\sim = [x]_{\mathcal{F}}$. $\mathbb{R}_* := \{{}^*x : x \in \mathbb{R}\}$

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For $x \in \mathbb{R}$, ${}^*x := (x, x, x, x, x, ...)/ \sim = [x]_{\mathcal{F}}$. $\mathbb{R}_* := \{{}^*x : x \in \mathbb{R}\}$
$\omega > x$ for all $x \in \mathbb{R}_*$. $\varepsilon < |x| \forall x \neq 0 \in \mathbb{R}_*$.

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These properties define "infinite" and "infinitesimal.

Extended functions and such

For $f : \mathbb{R} \to \mathbb{R}$, ${}^*f : \mathcal{R} \to \mathcal{R}$ is defined by
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If $[(x, x, x, x, x, \ldots)] \equiv y$, call x the **standard part** of y, $st(y)$.
e.g., $st(1, 1/2, 1/3, 1/4, 1/5, \ldots) = 0$.

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Riemann integration

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Ok, that's kind of dumb.

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Theorem

Tychonoff product theorem: Let X_j, $j \in J$, be compact topological spaces. Then $\prod_{j \in J} X_j$ is compact in the product topology.

Necessary background - superstructures

Let X be a set. (We will assume $\mathbb{N} \subseteq X$)
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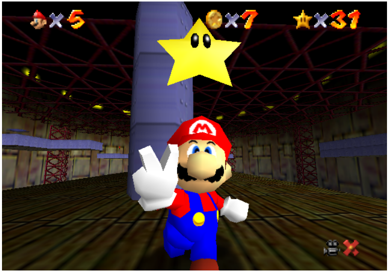
Will define for $\mathbb{R} \to \mathcal{R}$ and $\mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathcal{R})$ here.

star transforms

star transforms - here we go!



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$$\{[a, a, a, ...]_{\mathcal{F}}, [b, b, ...]_{\mathcal{F}}, [c, c, c, ...]_{\mathcal{F}}\}$$
$$*\mathbb{R} = \mathcal{R} \supsetneq \mathbb{R}_*$$

Properties

$A \subset B \Rightarrow *A \subset *B$
$a \in A \Rightarrow *a \in *A$
$*(A \cap B) = *A \cap *B$
$*(A \cup B) = *A \cup *B$
Generalizations have $*(A \times B) = *A \times *B$.

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Non-standard Archimedean principle: $\forall x \in *\mathbb{R} \exists n \in *\mathbb{N} : n > x$.
NOT: $\forall x \in *\mathbb{R} \exists n \in \mathbb{N} : n > x$. This is FALSE.

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Theorem

Robinson's theorem C is compact **iff** *each $y \in *A$ belongs to $m(x)$ for some $x \in A$.*

More equivalence...

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Robinson's Theorem does not depend on metrizability!

Tychonoff's Theorem

Theorem

Tychonoff product theorem: Let X_j, $j \in J$, be compact topological spaces. Then $X := \prod_{j\in J} X_j$ is compact in the product topology.

Proof.

Let $y \in *A$. There exist $x_j \in X_j$ such that $y_j \equiv x_j$, i.e., $y_j \in m(x_j)$. It follows that $y \in m(x)$. Thus X is compact.

□

The end.

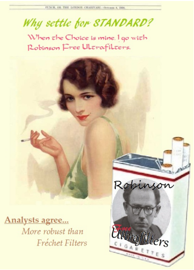

Figure: bondmatt.wordpress.com