

# A power estimate on the probability that Buffon's needle lands near a 1-dimensional Sierpinski gasket

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Recent joint work with Alexander Volberg

## Favard length/Buffon needle probability

Let  $E \subset \mathbb{C}$ , and let  $proj_\theta$  denote orthogonal projection onto the line forming the angle  $\theta$  with the real axis.

$$Fav(E) := Buf(E) := \frac{1}{\pi} \int_0^\pi |proj_\theta(E)| d\theta$$

The **Besicovitch projection theorem**: If  $H^1(E) < \infty$  and  $E$  is Besicovitch irregular, then  $Buf(E) = 0$

**Quantitative problem**: Let  $B_\varepsilon(E) = \{z : |z - E| < \varepsilon\}$ . Suppose  $E = \bigcap_{\varepsilon > 0} B_\varepsilon(E)$ . At what rate does  $Buf(B_\varepsilon(E)) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ ?

## Examples of sets

$\mathcal{K}$ , the 4-corner Cantor set with scaling factor  $1/4$ .  $\mathcal{K}_n := B_{4^{-n}}(\mathcal{K})$

$\mathcal{G}$ , the Sierpinski gasket with scaling factor  $1/3$ .  $\mathcal{G}_n := B_{3^{-n}}(\mathcal{G})$

$$\frac{\log n}{n} \lesssim \text{Buf}(\mathcal{K}_n) \lesssim \frac{1}{n^p}, \quad p < 1/6 \quad (1)$$

$$\frac{\log n}{n} \lesssim \text{Buf}(\mathcal{G}_n) \lesssim \frac{1}{n^p}, \quad p < 1/100? \quad (2)$$

Both lower bounds: Bateman and Volberg, 2008. One proof for both.

Upper bound of (1): Nazarov, Peres, and Volberg (2008).

Extended to some other product sets by Laba and Zhai (2009), with smaller  $p$ .

Upper bound of (2): Bond and Volberg (2009). Same approach as in (1), with some new difficulties.

## Setup and main ideas

$B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ . For  $\alpha \in \{-1, 0, 1\}^n$  let

$$z_\alpha := \sum_{k=1}^n \left(\frac{1}{3}\right)^k e^{i\pi[\frac{1}{2} + \frac{2}{3}\alpha_k]}$$

$$\text{Let } \mathcal{G}_n := \bigcup_{\alpha \in \{-1, 0, 1\}^n} B(z_\alpha, \frac{1}{2}3^{-n}).$$

$$\text{Let } f_{n,\theta} := \sum_{\alpha \in \{-1, 0, 1\}^n} \text{proj}_\theta(B(z_\alpha, \frac{1}{2}3^{-n}))$$

Fix  $\theta$ . If the mass of  $f_{n,\theta}$  is concentrated on a small set, then  $\|f_{n,\theta}\|_p$  should be large for  $p > 1$  - and vice versa.

**Lower bounds:**  $1 = \int f \leq \|f_{n,\theta}\|_p \|\chi_{\text{proj}_\theta(\mathcal{G}_n)}\|_q$ , so  $m(\text{proj}_\theta(\mathcal{G}_n)) \geq \|f\|_p^{-q}$ . A decent estimate.

**Upper bounds:**  $m(\text{proj}_\theta(\mathcal{G}_n)) \leq 1 - (K - 1)m(\{x : f_{n,\theta}(x) \geq K\})$ .

A pitiful estimate.

## Main idea, continued

For most directions, lots of stacking occurs. However, the pathological set of directions where  $|proj_\theta(\mathcal{G})| > 0$  is a **dense** countable set (Kenyon, 1997)! To control  $Buf(\mathcal{G}_n)$ , control the measure of the set of bad directions:

$$f_{N,\theta}^* := \max_{n \leq N} f_{n,\theta}(x)$$

$$A_{N,K,\theta} := \{x : f_{N,\theta}^*(x) \geq K\}$$

$$E_{N,K} := \{\theta : m(A_{N,K,\theta}) \leq 1/K^3\}$$

i.e.,  $E_{N,K}$  is the set of directions with not much triangle stacking. Fix  $\epsilon_0$  small enough, and let  $K = N^{\epsilon_0}$ .

### Theorem:

For all  $N \geq N^* \gg 1$ ,  $|E_{N,K}| < 1/K$ .

# Fourier setup

## Theorem:

$$E_N \subseteq \{\theta : \|f_{n,\theta}\|_2^2 \leq CK \quad \forall n \leq N\}.$$

$$f_{n,\theta} = \nu_n * 3^n \chi_{[-\frac{1}{2}3^{-n}, \frac{1}{2}3^{-n}]}, \text{ where } \nu_n := *_{k=1}^n \tilde{\nu}_k \text{ and}$$

$$\tilde{\nu}_k = \frac{1}{3} [\delta_{3^{-k} \cos(\pi/2-\theta)} + \delta_{3^{-k} \cos(-\pi/6-\theta)} + \delta_{3^{-k} \cos(7\pi/6-\theta)}].$$

(Point masses centered at  $\text{proj}_\theta(z_\alpha)$  convolved with intervals  $[-\frac{1}{2}3^{-n}, \frac{1}{2}3^{-n}]$ )

On the Fourier side:

$$\varphi_\theta(x) := \frac{1}{3} [e^{-ix \cos(\pi/2-\theta)} + e^{-ix \cos(-\pi/6-\theta)} + e^{-ix \cos(7\pi/6-\theta)}],$$

$$\widehat{\nu}_n(x) = \prod_{k=1}^n \varphi_\theta(3^{-k}x)$$

## Sample integral on the Fourier side

Let  $K = N^{\epsilon_0}$ , and let  $m = \epsilon_0 \log N$ . Let  $\tilde{E}_N \subset E_N$  such that  $|\tilde{E}| > |E/2|$  and let  $N/4 < n < N/2$  such that for all  $\theta \in \tilde{E}$ ,

$$\int_{3^{n-m}}^{3^n} \prod_{k=0}^n |\varphi_{\theta}(3^{-k}x)|^2 dx \leq \frac{2CKm}{N} \leq 2\epsilon_0 N^{\epsilon_0-1} \log N.$$

$n$  does not depend on  $\theta$ ;  $n$  can be extracted to satisfy the estimate in the average over  $\theta \in E$ , and then one chooses  $\tilde{E}$ .

This says that for  $\theta$  in a large set, a small sample integral of  $|\hat{f}_{n,\theta}|^2$  takes on a small value. Next, we remember the assumption that  $\tilde{E}$  is large, and prove a lower bound which can only be true for  $N < N^*$ .

(The lower bound is  $N^{-C\epsilon_0}$ .)

# Splitting $P$ into frequency groups

$\ell := \alpha m$ , recall  $m = \log K = \epsilon_0 \log N$

$$P_{1,\theta}^{\sharp}(z) = \prod_{k=1}^{n-m-\ell-1} \varphi_{\theta}(3^{-k}z)$$

$$P_{1,\theta}^{\flat}(z) = \prod_{k=n-m-\ell}^{n-m-1} \varphi_{\theta}(3^{-k}z)$$

$$P_{2,\theta}(z) = \prod_{k=n-m}^n \varphi_{\theta}(3^{-k}z)$$

High, medium, and low frequencies, respectively.

\*  $P_{2,\theta}$  has low frequencies, and therefore little regularity. We must control the damage caused by its small values.

\* For fixed  $\theta$ ,  $P_{1,\theta}^{\sharp}$  is well-behaved on small  $x$ -intervals.

\*  $P_{1,\theta}^{\flat}$  is approximately constant on small  $x$ -intervals, and is well-behaved in  $\theta$ .

## Estimate on the set of small values of $P_2$

$$SSV(\theta) := \{x : |P_{2,\theta}(x)| \leq 3^{-Cm}\}$$

For all  $\theta \in E$ ,  $\int_{3^{n-m}}^{3^n} |P_1^\sharp|^2 |P_1^b|^2 dx \gtrsim N^{\epsilon_0}$ . Show that this doesn't live on  $SSV$ .

$$\Omega := \frac{1}{|\tilde{E}|} \int_{\tilde{E}} \int_{SSV(\theta)} |P_1^\sharp|^2 |P_1^b|^2 dx d\theta$$

Analytically continue the real parts of the zeroes of  $P_2$ , and call them  $x_j$ . Pass from  $\tilde{E}$  to  $[0, \pi]$ .

$$\Omega \lesssim \sum_j \frac{1}{|\tilde{E}|} \int_0^\pi |P_{1,\theta}^b(x_j(\theta))|^2 \int_{I_j(\theta)} |P_{1,\theta}^\sharp|^2 dx d\theta$$

$$\Omega \ll N^{\epsilon_0}$$